Essay: Symmetry protected topological phases

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I. INTRODUCTION

The ability of machines to store and process information has seen a rapid progress over the course of the last 200 years. The exponentially increasing abilities of von Neumann architecture computers was predicted by Moore in 1965. Still, the task of simulating even simplified physical systems as small as lattices of 40 spins pose a challenge to classical computers. Moreover, Moore's law is approaching quantum limits [1]. Therefore, Richard Feynman proposed in 1982 the usage of computers whose architecture is composed of the very systems that scientists are so eager to simulate [2].

Very much like the development of now so called classical computers, the endeavour of creating quantum computing architectures requires reliable building blocks for storing and processing quantum information. Different approaches are competing for the best decoherence rates and gate fidelities such as trapped ions, lattices of neutral atoms, photonic systems, superconducting computers in research and industry [3].

As decoherence times are often a compromise between good accessibility and system isolation, it makes sense to think of ways of how to protect states more creatively. Everyone who owns headphones knows how stable a knot in a cord is under the physical interactions with the environment in their pocket. This principle was already used by andean South American cultures in 3000 BC by encoding information in Quipu knots [4]. The idea of symmetry protected topological phases can be understood in a similar manner: Through the topological properties of the phase some properties are conserved under environment effects.

A particular symmetry protected topological state is the Majorana Fermion. It is its own antiparticle and was first introduced by Ettore Majorana in 1937 [5]. To this date no elementary particle Majorana Fermions were found.

With the Majorana Fermion being fermionic states, qubits can be encoded. This is useful as they are protected from local perturbations unlike charge qubits. More precisely: Quantum states are sensitive to population and phase errors. Classical errors can be prevented by separating the charges spatially. The phase error becomes unlikely as a result.

Ten years later after Kitaevs proposal, hope for experimental evidence arose when Stanescu et al. promoted the idea to look at zero energy peaks [6]. One year later Mourik et al. at Delft proposed a signature of Majorana Fermions [7].

This essay is focused around the idea of unpaired Majorana Fermions in quantum wires as proposed by Kitaev in 2001 [8], and experimental signatures of Majorana Fermions in Hybrid Superconductor-Semiconductor Nanowire Devices discovered by Mourik et al. in 2012 [7]. After a brief motivation, the theoretical background of Kitaev's toy model is presented, afterwards the findings of Mourik et al. will be discussed. Finally, an outlook is given on the future of Majorana Fermions.

II. KITAEV'S TOY MODEL OF MAJORANA FERMIONS

The motivation for this section is to show Kitaev's way to achieve isolated Majorana Fermions.

A. Preliminary definitions

As Majorana Fermions are their own antiparticles we need to construct creation and annihilation operators that are identical. We make use of the complex Fermion operatos $c^\dagger$ and c:

$$\begin{aligned} \{\hat{c}_i^\dagger, \hat{c}_j\} &= \delta_{ij} & \hat{c}^2 &= (\hat{c}^\dagger)^2 = 0 \\ \hat{c}^\dagger |0\rangle &= |1\rangle & \hat{c}^\dagger |1\rangle &= 0 \\ \hat{c}\,|0\rangle &= 0 & \hat{c}\,|1\rangle &= |0\rangle \end{aligned}$$

Majorana Fermions $\hat{\gamma}_j$ correspond to the real and the imaginary part of a complex fermion.

$$\hat{\gamma}_1 = \hat{c}^\dagger + \hat{c} \tag{1}$$

$$\hat{\gamma}_2 = \frac{1}{i}\left(\hat{c}^\dagger - \hat{c}\right) \tag{2}$$

with $\{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij}$ and $\hat{\gamma}^\dagger = \hat{\gamma}$. We can see now that Majorana Fermions are their own antiparticle: $\hat{\gamma}^2 = 1$. Every fermion is expressed by a pair of Majorana Fermions. Therefore, they have to appear in an even number. In order to achieve long decoherence times we need to seperate them spatially.

B. Formulating the Hamiltonian

Here, a model is chosen that has seperated states as ground states. As a first step we formulate the Hamiltonian as a dimerized chain of Majorana Fermions:

$$\hat{H} = \underbrace{i\sum_{j=1}^{L-1} t\hat{\gamma}_{2j}\hat{\gamma}_{2j+1}}_{\text{coupling between unit cells}} + \underbrace{i\sum_{j=1}^{L} \mu\hat{\gamma}_{2j-1}\hat{\gamma}_{2j}}_{\text{coupling within unit cells}} \tag{3}$$

with $\hat{\gamma}_{2j-1} = \hat{c}_j + \hat{c}_j$, $\hat{\gamma}_{2j}$. Of course, the magnitude of t and μ indicate which interaction dominates.

We notice that $\hat{H}$ has a parity symmetry: It commutes with

$$\hat{P} = \prod_{j=1}^{L} -i\left[\hat{\gamma}_{2j-1}\hat{\gamma_{2j}}\right]. \tag{4}$$

This represents a $\mathbb{Z}_2$ symmetry.

C. Finding the ground state

Let $t = 1$ and $\mu = 0$, then the coupling between unit cells dominates, leading to two spatially seperated Majorana Fermions, which can be thought of as the real and imaginary parts of complex fermions. The possible ground state

$$\hat{f} = \frac{1}{2}\left(\hat{\gamma}_1 + i\hat{\gamma}_L\right) \tag{5}$$

has a twofold degeneracy of $|0\rangle$ or $\hat{f}^\dagger|0\rangle$.

We can rewrite the Hamiltonian using complex fermions:

$$\hat{H} = -t\sum_{j=1}^{L-1}\left[\hat{c}^\dagger_{j+1}\hat{c}_j + \hat{c}^\dagger_{j+1}\hat{c}^\dagger_j + \text{h.c.}\right] - 2\mu\sum_{j=1}^{L}\hat{c}^\dagger_j\hat{c}_j \tag{6}$$

The ground state for strong cell interaction $t = 1$, $\mu = 0$ can be either even or odd:

$$|\psi\rangle \approx \sum_{even/odd} |n_1 n_2 \ldots n_L\rangle. \tag{7}$$

The Hamiltonian 6 is similar to the 1D quantum Ising model. After performing several transformations we can write it in a Bogoliubov basis, and recall the quantum phase transition at $\mu = t$. Finally, we arrive at:

$$\hat{H} = \sum_k \epsilon_k \hat{d}^\dagger_k \hat{d}_k, \tag{8}$$

$$\epsilon_k = \pm 2\sqrt{(t\cos(k-\mu)^2 + t^2\sin^2(k))} \tag{9}$$

Here, the Hamiltonian resembles the interaction between the Majorana pairs at the edges of a quantum wire. The dispersion relation describes the energy momentum relation.

D. A closer look on boundary conditions

The Majorana chain has a degenerate ground state only for open boundaries while the two-fold degeneracy of the Ising chain is insensitive to the boundary condition.

Now we consider a closed boundary condition: One of the two sectors is frustrated and the ground state energy is $2|t|$ higher. The reason for this is that the coupling term between the last and first site has a different sign in the Jordan Wigner string transformation depending on the parity eigenvalue.

$$\hat{H} = -t\sum_{j=1}^{L-1}\hat{\sigma}^2_j\hat{\sigma}^2_j \underset{\text{even/odd sector}}{\pm t\hat{\sigma}^2_L\hat{\sigma}^2_1} \tag{10}$$

E. Stability of edge modes

To analyze the stability of the edge modes, we rewrite the Majorana Hamiltonian:

$$\hat{H} = \frac{i}{2}\sum A_{ij}\hat{\gamma}_i\hat{\gamma}_j \tag{11}$$

$$\text{with} \quad A = \begin{pmatrix} 0 & \mu & & & \\ -\mu & 0 & t & & \\ & -t & 0 & \mu & \\ & & -\mu & \ddots & \mu \\ & & & -\mu & 0 \end{pmatrix} \tag{12}$$

If Matrix A has a zero energy eigenstate, we know that the particle eigenstates ϵ_n are degenerate.

Now a state that is an eigenstate of A with zero energy is constructed in the following manner:

$$u^l = (1, 0, \frac{\mu}{t}, 0, \frac{\mu}{t}^2, 0, \ldots). \tag{13}$$

Also an edge mode operator is constructed as $\hat{b} = \sum u^{l/r}_j \hat{\gamma}_j$ that is exponentially localized at the edges for $|\frac{\mu}{t} \leq 1|$.

Now, the coupling between the two modes at the edges lifts the degeneracy:

$$H_{\text{eff}} \approx e^{-\frac{L}{\xi}}\hat{b}^l\hat{b}^r, \quad \xi = 1/\ln(\frac{t}{\mu}) \tag{14}$$

With ξ being the correlation length.

In conclusion edge modes are robust as long as $L \gg \xi$ and parity symmetry is required. Also, we have to remove the spin degeneracy as the local term would bond otherwise. Therefore, we would need spinless fermions in theory.

F. Conditions for Majorana Fermions

A finite Majorana chain of length L possesses two ground states with an energy difference proportional to

$e^{-\frac{L}{\xi}}$ and different fermionic parities. As those states are through spatial separation immune to decoherence and have a fermionic character, they can be used as qubits.

The property of a system to have boundary Majorana Fermions is expressed as a condition on the bulk electron spectrum.

G. Hints to physical realizations

Looking at the standard model, we can not identify suitable particles that resemble the needed properties of Majorana Fermions. Therefore, we have to retreat to quasiparticles in condensed matter.

A hint can be obtained by looking at the fermionic operators. The combination of creation and annihilation operators hints at the possiblity that electron-hole excitons could be a way of creating Majorana Fermions. But in fact, both electron and hole being fermions the whole quasiparticle would be one boson.

Therefore, instead of a bound state we need a superposition state. A natural place to look for that are in superconductors. The reason lies in the condensate of cooper pairs: Intuitively, one can start with a hole and add a cooper pair, in sum resulting in roughly one electron.

Another problem lies in the zero point motion just like in a harmonic oscillator. This issue can be overcome with p-wave superconductors as the zero motion can be cancelled by the π Berry-Phase.

Kitaev proposed that all conditions are satisfied in the presence of an arbitrary small energy gap induced by the proximity of a three-dimensional p-wave superconductor, provided that the normal spectrum has an odd number of Fermi points in each half of the Brillouin zone. An alternative lies in 1D s-wave superconductors. As the spin degeneracy has to be lifted, the usage of a magnetic field is plausible.

These ideas for the implementation of quantum wires inspired experimental groups for the search of signatures of Majorana Fermions.

III. SIGNATURES OF MAJORANA FERMIONS IN HYBRID SUPERCONDUCTOR-SEMICONDUCTOR NANOWIRE DEVICES

In this section evidence by Mourik et al. is presented that supports the hypothesis of Majorana Fermions in nanowires that are coupled to superconductors.

A. Methods for experimental realizations of Majorana Fermions

Let us begin with a simple one particle model that represents all major ingredients that we need according to Oreg et al. [9].

$$\hat{H} = \underbrace{\left(p^2/2m - \mu(y)\right)\tau_z}_{E_{\text{kin}},\text{ chem pot.}} + \underbrace{u(y)p\sigma_z\tau_z}_{\text{spin-orb. Int.}} + \underbrace{B(y)\sigma_x}_{\text{Zeeman F.}} + \underbrace{\Delta(y)\tau_x}_{\text{SC}} \tag{15}$$

Here, the following question arises: When does the topological transition ocurr? In fact, spin-orbit interaction leads to degeneracy along k and the chemical potential μ and magnetic field B can be tuned to drive topological phase transition. As a result a coupling in the s-wave superconductor emerges. Finally, Majorana Fermions should arise as zero-energy bound states, one at each end of the wire.

The resulting recipe can be summarized as follows: Start with a conventional superconductor and glue a 1D nano wire of a material with a large spin-orbit coupling on it. Then, a magnetic field is applied as spinless fermions are required. In the following, we will look at Mourik et al.'s realization of that recipe and the resulting observations.

B. Electrical measurements on nanowires

Mourik et al. use InSb (indium antimonide) for the nanowire, as it has a large g factor and large spin-orbit coupling. In the experiment it sits on a s-wave superconductor (niobium titanium nitride). Electrical measurements are performed via nanowires that are contacted with one gold and one superconducting electrode.

C. Regulating electron density and the tunnel barrier through gate voltage

Through gate voltages the electron density is varied and the tunnel barrier between normal and superconducting contacts is defined. A tunnel barrier in the nanowire is realized by applying a negative voltage. The differential conductance dI/dV at voltage V and current I is proportional to the density of states at energy $E = eV$. With an induced gap of 250 μeV in the superconductor, a finite dI/dV is measured. Whereas points of resonances imply Andreev bound states, nonresonant current indicates a not fully developed proximity gap. Finally, at $V = 0$ and a magnetic field between $100 - 400$ mT zero bias peaks arise.

D. Variation of the magnetic field and voltage after zero bias voltage

To inspect how stable they are, the magnetic field is varied between -0.5 to 1 T. Most interestingly, the sweep directions show a symmetry around the $B = 0$ point. This indicates the absence of hysteresis.

The bound states also remain fixed to zero bias when the gate voltages are changed.

E. Verification of spin-orbit coupling

In theory, the external magnetic field must have a non-zero perpendicular component in order for a zero bias peak to appear. A 3D vector magnet is used to sweep arbitrary directions for the magnetic field. Here, Mourik et al. found that the zero bias peak is absent around $\pi/2$, which should be the direction of the spin orbit magnetic field. In extent to that the zero bias peak appears for any angle of the magnetic field that is perpendicular to the spin orbit magnetic field.

F. Main observations

- The zero bias peak exists over a large voltage range.
- The zero bias peak can be split in two peaks located symmetrically around zero.
- The peak can not be moved away from zero to finite bias.
- The effect of spin-orbit interaction is evident as the zero bias peak comes and goes with the angle of the B field with respect to the wire axis.

These observations support the hypothesis of Majorana Fermions in nanowires coupled to superconductors.

IV. SUMMARY

Now, a brief summary of Kitaev and Mourik et al.'s findings is presented:

A finite Majorana chain of length L possesses two ground states with an energy difference proportional to $e^{-\frac{L}{\xi}}$ and different fermionic parities. As those states are through spatial separation immune to decoherence and have a fermionic character, they can be used as qubits.

The property of a system to have boundary Majorana Fermions is expressed as a condition on the bulk electron spectrum.

Kitaev proposed that all conditions are satisfied in the presence of an arbitrary small energy gap induced by the proximity of a three-dimensional p-wave superconductor.

Mourik et al. observed a zero bias peak in three different devices and in two different setups. A zero bias peak appears at finite B and sticks to zero bias over a range from 0.07 to 1 T. Furthermore, it remains at zero bias while changing the voltage on any of the gates over large ranges. The effect of spin-orbit interaction is evident as the zero bias peak comes and goes with the angle of the B field with respect to the wire axis. Finally, the rigid zero bias peak is absent when the superconductor is replaced by a normal conductor.

Mourik et al. conclude that this provides evidence for the existence of Majorana Fermions. However, they also confess in the paper that they did not address the topological properties of Majorana Fermions as they don't analyze conductance quantization. Of course there are other possibilities that justify the observations rather than Majorana Fermions.

Ultimately, only braiding experiments can reveal the non-Abelian statistics of the Majorana Fermions.

V. OUTLOOK

Much has happened since Mourik et al.'s experiment: The first prototype for a Qubit with Majoranas was realized by Aasen et al. in 2016 by using semiconductor wires with mesoscopic superconducting islands [10]. As with any technology that realized physical qubits the key question remains: How well can we scale it?

This question was adressed by Karzig et al., who proposed a scalable quantum computer composed of qubits encoded in aggregates of four or more Majorana zero modes [11]. This method could be executed by improving lithographic techniques by intelligent 2D design by the ManfraLab Station Q Purdue.

In the future we might be able to use quantum computers to eludicate reaction mechanism that are costly to simulate on classical computers [12]. It remains an open question which hardware implementation of qubits will win the race of balancing fidelity and scaling. Just like headphone cables will take a long time to entangle, Majorana Fermions will remain a prominent competitor due to their outstanding symmetry protected coherence times.

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